Thermodynamic derivation of the mutual information for discrete symmetric channels

2008 J. Phys. A: Math. Theor. 41 402002

(http://iopscience.iop.org/1751-8121/41/40/402002)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 402002 (7pp) doi:10.1088/1751-8113/41/40/402002

FAST TRACK COMMUNICATION

Thermodynamic derivation of the mutual information for discrete symmetric channels

Ido Kanter[1], **Ori Shental**[2], **Hadar Efraim**[1] **and Nadav Yacov**[1]

[1] Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel
[2] Center for Magnetic Recording Research, University of California, San Diego, 9500 Gilman Drive, La Jolla, CA 92093, USA

Received 24 July 2008, in final form 23 August 2008
Published 15 September 2008
Online at stacks.iop.org/JPhysA/41/402002

Abstract
The mutual information for the binary symmetric channel as well as for the discrete symmetric channel consisting of 4-input/output (I/O) symbols is explicitly calculated using the generalized second law of thermodynamics which was recently proposed by the first two authors. For equiprobable I/O the mutual information of the examined channels has a very simple thermodynamic form as a function of the internal energy of the channel. We prove that this simple form of the mutual information governs the class of discrete memoryless symmetric communication channels with equiprobable I/O symbols.

PACS numbers: 05.70.−a, 89.70.−a, 89.70.Cf, 89.70Kn

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The current scientific conception is that the theory of information is a creature of mathematics and has its own vitality independent of the physical laws of nature [1]. The first two authors have recently proved [2, 3] that the principal quantity in the theory of information, the mutual information, can be reformulated as a consequence of the fundamental laws of nature—the laws of thermodynamics. This corollary was originally exemplified for the Gaussian noisy channel.

The generic problem in information processing is the transmission of information over a noisy channel [4–6]. This central paradigm of information theory can be mathematically abstracted for having two random variables X and Y representing the desired information and its noisy replica, respectively. Noisy transmission can occur either via space from one geographical point to another, as happens in communications, or in time, for example, when sequentially writing and reading files from a hard disk in the computer.

Mutual information, $I(X;Y)$, quantifies the amount of information in common between two random variables and it is used to upper bound the attainable rate of information transferred across a channel. To put it differently, mutual information measures the amount of information that can be obtained about one random variable (channel input X) by observing another (output Y). A basic property of the mutual information is that $I(X;Y) = H(X) - H(X|Y)$, where $H(\cdot)$ is the information (Shannon) entropy [1]. It measures the amount of uncertainty in a random variable, indicating how easily data can be losslessly compressed. Hence knowing Y, we can save an average of $I(X;Y)$ bits in encoding X compared to not knowing Y.

As recently shown by the first two authors [2, 3], the modeling of the Gaussian channel as a thermal system requires the generalization of thermodynamics to include temperature-dependent Hamiltonians, as well as the redefinition of the notion of temperature. The generalized second thermodynamic law was proven to have the following form:

$$\mathrm{d}S = \frac{\mathrm{d}Q}{T} - \frac{1}{T}\left\langle \frac{\mathrm{d}\mathcal{E}(X)}{\mathrm{d}T} \right\rangle \mathrm{d}T, \tag{1}$$

where $\langle\cdot\rangle$ denotes averaging over the Boltzmann distribution.

The generalized second law of thermodynamics (1) has a clear physical interpretation. For simplicity, let us assume that an examined system is characterized by a comb of discrete energy levels $\mathcal{E}1, \mathcal{E}2, \ldots$, see figure 1(*a*) . The heat absorbed into the T-dependent system has the following dual effect: a first contribution of the heat, $\mathrm{d}U - \langle \mathrm{d}\mathcal{E}(X)/\mathrm{d}T\rangle \mathrm{d}T$, increases the temperature of the system, figure 1(*b*), while the second contribution, $\langle \mathrm{d}\mathcal{E}(X)/\mathrm{d}T\rangle \mathrm{d}T$, goes for shifting the energy comb, figure 1(*c*). However, the shift of the energy comb does *not* affect the entropy, since the occupation of each energy level remains the same, and the entropy is independent of the energy values which stand behind the labels $\mathcal{E}1, \mathcal{E}2, \ldots$. The change in the entropy can be done only by moving part of the occupation of one tooth of the energy comb to the neighboring teeth, figure 1(*b*). Hence, the *effective heat* contributing to the entropy is $\mathrm{d}Q - \langle \mathrm{d}\mathcal{E}(X)/\mathrm{d}T\rangle \mathrm{d}T$, and this is the physical explanation to the generalized second law (1). A schematic picture of the communication heat engine is depicted in figure 2, where the heat is devoted to the change of the Hamiltonian (without altering the thermodynamic entropy) denoted by the term 'working channel'. Note that for T-independent Hamiltonians the traditional picture of the heat engine is recovered as well as the traditional second thermodynamic law [7–9].

Using the generalized second law of thermodynamics, (1), one can show [2, 3] that the generalized expression for the mutual information is given by

$$I = -\gamma U(\gamma)|_0^\beta + \left\langle \int_0^\beta \left(U(\gamma, y) + \gamma \left\langle \frac{\mathrm{d}E}{\mathrm{d}\gamma} \right\rangle_{x|y} \right) \mathrm{d}\gamma \right\rangle_y . \tag{2}$$

Throughout this contribution the expectation operation $\langle\cdot\rangle$ is taken with respect to its subscript. Note that this thermodynamic expression for the mutual information holds for any channel which can be described by a thermal system exhibiting quasi-static heat transfer. For the Gaussian channel with a standard Gaussian input and signal-to-noise ratio, snr, one can show [2, 3] that the celebrated formula for the Shannon capacity [1] is obtained from (2), $I(X;Y) = \frac{1}{2}\log(1+\beta)$, where $\beta = \mathsf{snr}$.

2. Binary symmetric channel

We now turn to derive the mutual information for the archetypal discrete memoryless channel, the binary symmetric channel (BSC), with input x and output y. The input's prior distribution obeys $P(x = 1) = P(x = -1) = 1/2$ and the probability for a symbol to flip during the transmission is denoted by δ, $P(y = \pm|x = \mp) = \delta$, see figure 3. Hence, the conditional

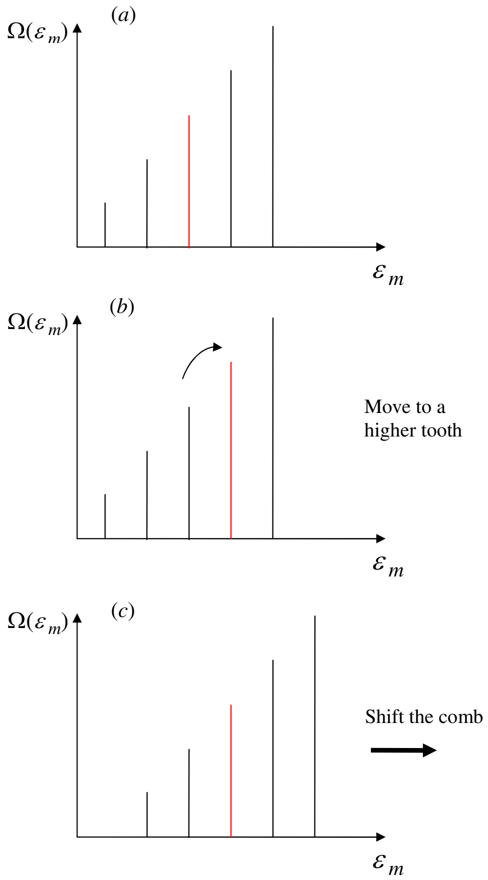

Figure 1. A system consisting of a discrete energy comb (levels), ϵ_m, with the corresponding degeneracy $\Omega(\epsilon_m)$ which increases with the energy. For simplicity of presentation we assume that the system is occupying only one tooth of the comb, depicted by a red tooth in (*a*). As heat is absorbed into the system, the system can either increase the temperature by jumping to the next tooth of the comb, (*b*), or shift the energy comb (*c*). Note that only the jump to the next tooth, (*b*), changes the entropy of the system, where in (*c*) the entropy remains the same as in (*a*).

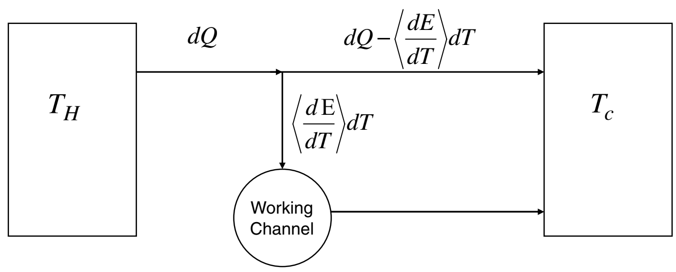

Figure 2. A schematic communication heat engine. H/C denote hot/cold temperatures.

probability of the output given the input is

$$P(y|x) = \delta^{\frac{1-xy}{2}}(1-\delta)^{\frac{1+xy}{2}} = \exp\left[\frac{xy}{2}\ln\left(\frac{1-\delta}{\delta}\right) + \frac{1}{2}\ln(\delta(1-\delta))\right]. \qquad (3)$$

$x \backslash y$	+	-
+	$1-\delta$	δ
-	δ	$1-\delta$

Figure 3. The input/output transition probabilities for the binary symmetric channel.

A comparison of the channel's *a posteriori* probability distribution, given by Bayes' law

$$P(X = x|Y = y) \propto P(Y = y|X = x) \tag{4}$$

with the Boltzmann distribution law yields the following mapping of the energy and the inverse temperature, β, of the equivalent thermal system

$$E = -\frac{xy}{2} - \frac{1}{2\beta}\ln(\delta(1-\delta)), \qquad \beta = \ln\frac{1-\delta}{\delta}. \tag{5}$$

Note that since the inverse temperature, $\beta(\delta)$, is positive, then $\delta = 1/(1+\exp(\beta)) < 1/2$ [10, 11]. This definition of δ coincides with two limiting cases. For $\delta = 1/2$, $\beta = 0 (T \to \infty)$, where for $\delta = 0$, $\beta \to \infty$ $(T = 0)$. Using $\langle x\rangle_{x|y} = y(1-2\delta)$, one can find that the internal energy is given by

$$U(\beta) = -\frac{1-2\delta}{2} - \frac{1}{2} + \frac{\ln(1+\exp(\beta))}{\beta}. \tag{6}$$

Note that for the BSC specifically the internal energy is independent of y due to the binary nature of the input/output (I/O) symbols. Similarly to the internal energy on can find

$$\left\langle \beta\frac{\mathrm{d}E}{\mathrm{d}\beta}\right\rangle_{x|y} = -\frac{\ln(1+\exp(\beta))}{\beta} + \frac{\exp(\beta)}{1+\exp(\beta)}. \tag{7}$$

It is now easy to verify that the second term of the generalized mutual information, (2), vanishes,

$$U(\beta) + \left\langle \beta\frac{\mathrm{d}E}{\mathrm{d}\beta}\right\rangle_{x|y} = 0, \tag{8}$$

and the mutual information for the equiprobable I/O BSC (which is also the Shannon capacity of the BSC in general) has a very simple form given explicitly (in nats) by

$$\begin{aligned} I &= -\gamma U(\gamma)|_0^\beta = \beta(1-\delta) - \ln(1+\exp(\beta)) + \ln 2 \\ &= (1-\delta)\ln(1-\delta) + \delta\ln(\delta) + \ln 2 = \ln 2 - H_B(\delta), \end{aligned} \tag{9}$$

where $H_B(\cdot)$ denotes the binary entropy. Hence, the mutual information for the BSC is recovered using the generalized second thermodynamic law. Note that in contrast to the Gaussian channel, in the BSC case the second term of the mutual information, (2), vanishes, and the mutual information is proportional to the internal energy

$$I = -\gamma U(\gamma)|_0^\beta. \tag{10}$$

x_1, x_2 \ y_1, y_2	+ +	+ -	- +	- -
+ +	$1-\delta$	$\varepsilon\delta$	$\varepsilon\delta$	$(1-2\,\varepsilon)\delta$
+ -	$\varepsilon\delta$	$1-\delta$	$(1-2\,\varepsilon)\delta$	$\varepsilon\delta$
- +	$\varepsilon\delta$	$(1-2\,\varepsilon)\delta$	$1-\delta$	$\varepsilon\delta$
- -	$(1-2\,\varepsilon)\delta$	$\varepsilon\delta$	$\varepsilon\delta$	$1-\delta$

Figure 4. The input/output transition probabilities for a channel of 4-I/O symbols.

3. 4-I/O symbols

Is this simple thermodynamic form, (10), a coincidence of the BSC only or does it occur for the general class of discrete memoryless channels? To find the answer to this interesting question, we first turn to discuss in detail the case of 4-I/O symbols. The input is represented by two binary units x_1 and x_2, and similarly the output units are y_1 and y_2. We assume that the 4-input symbols are equiprobable. The conditional probability of $P(y_1, y_2|x_1, x_2)$ obeys the following symmetry. The probability that both output units are equal to the input units, respectively, is $1-\delta$, the probability that only one unit is equal is $\epsilon\delta$ ($\epsilon \leqslant 1$), and the probability that the two output units differ from the input is $\delta(1-2\epsilon)$ (figure 4). Hence,

$$P(y|x) = (1-\delta)^{\frac{1+x_1y_1}{2}\frac{1+x_2y_2}{2}}(\epsilon\delta)^{\frac{1+x_1y_1}{2}\frac{1-x_2y_2}{2}}(\epsilon\delta)^{\frac{1-x_1y_1}{2}\frac{1+x_2y_2}{2}}(\delta-2\epsilon\delta)^{\frac{1-x_1y_1}{2}\frac{1-x_2y_2}{2}}, \tag{11}$$

and it is now easy to verify that

$$P(X=x|Y=y) \propto \exp\left\{\frac{x_1y_1+x_2y_2}{4}\ln\left(\frac{1-\delta}{\delta(1-2\epsilon)}\right)+\frac{1}{4}\ln((1-\delta)\delta^3\epsilon^2(1-\epsilon)) + \frac{x_1x_2y_1y_2}{4}\ln\left[\frac{1-\delta}{\delta}\frac{(1-2\epsilon)}{\epsilon^2}\right]\right\}.$$

Similarly to (5) one can find in this case

$$E = -\frac{1}{4}(x_1y_1+x_2y_2+x_1x_2y_1y_2)+\frac{1}{4\beta}\ln((1-\delta)\delta^3)+\frac{1}{4\beta}f(x_1y_1, x_2y_2, \epsilon), \tag{12}$$

where

$$f \triangleq (x_1y_1+x_2y_2)\ln(1-2\epsilon) - x_1x_2y_1y_2\ln\left(\frac{1-2\epsilon}{\epsilon^2}\right) - \ln(\epsilon^2(1-2\epsilon)), \tag{13}$$

$$\beta = \ln\left(\frac{1-\delta}{\delta}\right), \tag{14}$$

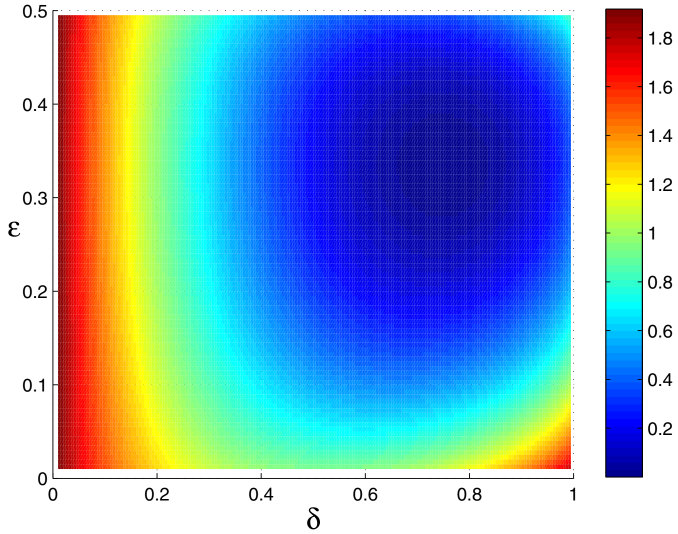

Figure 5. The mutual information as a function of δ and ϵ for the discussed case of 4-I/O symbols.

and the two limiting cases are $\delta = 3/4$ and $\epsilon = 1/3$ ($\beta = 0$) and $\delta = 0$ ($\beta \to \infty$). Using the following conditional expectations,

$$\langle x_m \rangle_{x_m|y_m} = y_m(1 - 2\delta + 2\epsilon\delta) = y_m(1 - 2\delta(1-\epsilon)), \qquad \langle x_1 x_2 \rangle_{x_1x_2|y_1y_2} = y_1 y_2 (1 - 4\epsilon\delta), \tag{15}$$

one can find that the internal energy

$$U(\beta) = -\frac{3}{4} + \delta - \frac{\ln((1-\delta)\delta^3)}{4\beta} - \frac{\delta}{\beta}[2\epsilon\ln(\epsilon) + (1-2\epsilon)\ln(1-2\epsilon)], \tag{16}$$

and again verify that

$$U(\beta) + \beta\left\langle \frac{\mathrm{d}E}{\mathrm{d}\beta}\right\rangle_{x|y} = 0. \tag{17}$$

Hence the mutual information has again the simple form, (10), and is given explicitly by

$$I = 2\ln 2 + (1-\delta)\ln(1-\delta) + 2\epsilon\delta\ln(\epsilon\delta) + \delta(1-2\epsilon)\ln((1-2\epsilon)\,\delta), \tag{18}$$

which can be verified (by direct computation of the mutual information out of its definition) to have the correct form. The mutual information as a function of (δ, ϵ) is depicted in figure 5, where the mutual information $I = 0$ for $\delta = 3/4$ and $\epsilon = 1/3$.

4. Equiprobable discrete memoryless symmetric channels

For the general case of discrete memoryless symmetric channels with 2^n equiprobable input symbols, the conditional probabilities have the following form:

$$P(y_1, \ldots, y_n | x_1, \ldots, x_n) = \delta\epsilon(x_1y_1, \ldots, x_ny_n), \tag{19}$$

and one can verify that similarly to (5) and (12) the energy is given now by

$$E = -\frac{1}{2^n}\left[-1 + \Pi_{k=1}^{n}(1 + x_k y_k)\right] - \frac{1}{2^n\beta}\ln((1-\delta)\,\delta^{2^n-1}) + \frac{1}{\beta} f_n(\{x_i y_i\}, \{\epsilon(x_1y_1, \ldots, x_ny_n)\}), \tag{20}$$

where f_n is a function of the number of symbols only. Now it is clear that

$$2^n \left(U(\beta) + \left\langle \beta \frac{\mathrm{d}E}{\mathrm{d}\beta} \right\rangle_{x|y} \right) = -\langle -1 + \Pi_{k=1}^n (1 + x_k y_k) \rangle_{x_k|y_k} - \frac{\mathrm{d}\ln((1-\delta)\delta^{2^n-1})}{\mathrm{d}\delta} \frac{\mathrm{d}\delta}{\mathrm{d}\beta},$$

where as in (5) and (12) $\beta = \ln((1-\delta)/\delta)$ and $\mathrm{d}\delta/\mathrm{d}\beta = -\delta(1-\delta)$. Hence

$$\begin{aligned} 2^n \left(U(\beta) + \left\langle \beta \frac{\mathrm{d}E}{\mathrm{d}\beta} \right\rangle_{x|y} \right) &= -\langle -1 + \Pi_{k=1}^n (1 + x_k y_k) \rangle_{x_k|y_k} + 2^n - 1 - 2^n \delta \\ &= -\langle \Pi_{k=1}^n (1 + x_k y_k) \rangle_{x_k|y_k} + 2^n (1-\delta) = 0. \end{aligned} \tag{21}$$

The identity to 0 is a result of $\langle \Pi_{k=1}^n (1 + x_k y_k) \rangle_{x_k|y_k}$ which is equal to the trace of the conditional probability, (19). Hence we proved that for a general memoryless symmetric channel consisting of 2^n equiprobable I/O symbols

$$I = -\gamma U(\gamma)|_0^\beta, \qquad U(\beta) + \left\langle \beta \frac{\mathrm{d}E}{\mathrm{d}\beta} \right\rangle_{x|y} = 0. \tag{22}$$

Note that the identity $I = -\gamma U(\gamma)|_0^\beta$ is in contrast to the case of a Gaussian channel with Bernoulli-1/2 or Gaussian inputs [2, 3].

5. Conclusion

In this contribution it is shown that the mutual information for the class of discrete memoryless symmetric communication channels can be explicitly derived via thermodynamic argumentation and a recent generalization of the second law of thermodynamics for temperature-dependent energy functions. For equiprobable I/O the mutual information of the examined discrete channels gets a very simple thermodynamic form, $-\gamma U(\gamma)|_0^\beta$, providing an intriguing relation between information theory and thermodynamics.

References

[1] Shannon C E 1948 *Bell Syst. Tech. J.* **27** 379–423 and 623
[2] Shental O and Kanter I 2008 arXiv:0806.3133v1
[3] Shental O and Kanter I 2008 arXiv:0806.3763v1
[4] Cover T M and Thomas J A 1991 *Elements of Information Theory* (New York: Wiley)
[5] Blahut R E 1987 *Principles and Practice of Information Theory* (Reading, MA: Addison-Wesley)
[6] Gallager R G 1968 *Information Theory and Reliable Communication* (New York: Wiley)
[7] Carnot S 1824 Réflexions sur la puissance motrice du feu et sur les machines propres à développer cette puissance
[8] Reif F 1965 *Fundamentals of Statistical and Thermal Physics* (New York: Mcgraw-Hill)
[9] Reichl L E 1980 *A Modern Course in Statistical Physics* (Austin, TX: University of Texas Press)
[10] Ruján P 1993 *Phys. Rev. Lett.* **70** 2968
[11] Nishimori H 2001 *Statistical Physics of Spin Glasses and Information Processing* (Oxford: Oxford University Press)